Suizumura-consistent relations: an overview*

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This version: May 5, 2017

Abstract. This paper provides a brief introduction to the use and usefulness of Suizumura consistency, a coherence requirement for binary relations that weakens transitivity. The property is introduced by Suizumura (1976b) in the context of collective choice but, as demonstrated in some recent contributions, its applicability reaches beyond the boundaries of social-choice theory. In addition to a summary of its mathematical underpinnings, some recent applications in individual and collective decision-making are provided. Several examples are employed to illustrate the property and how it distinguishes itself from alternative weakenings of transitivity such as quasi-transitivity or acyclicity. Journal of Economic Literature Classification Nos.: D01, D63, D71.

Keywords: Suizumura Consistency, Individual and Collective Choice.

* This paper is dedicated to Kotaro Suzumura in deep appreciation of his countless fundamental contributions to the academic community in general and to the members of the economics profession in particular. I thank a referee for thoughtful comments and suggestions. Financial support through grants from the Fonds de Recherche sur la Société et la Culture of Québec and the Social Sciences and Humanities Research Council of Canada is gratefully acknowledged.
1 Introduction

The notion of Suzumura consistency originates in a seminal contribution by Suzumura (1976b). This property weakens the well-established transitivity postulate that is ubiquitous in economic theory. The main purpose of this paper is to illustrate the use and the usefulness of this fundamental coherence property. In addition to an explanation of the condition and its mathematical underpinnings, some recent approaches that make use of it are summarized. No proofs are presented because they can easily be found in the original publications. Instead, I hope to succeed in providing a unified perspective to illustrate the potential of Suzumura consistency as a general tool that is not restricted to the few specific examples for applications discussed here.

First and foremost, Suzumura consistency can be used to obtain a substantial generalization of a classical result by Szpilrajn (1930). Szpilrajn’s extension theorem proves that any transitive relation can be extended to a complete and transitive relation. While the original version of the result is stated for strict relations, it can also be phrased in a setting where any two objects may be equally good. Formulated in this manner, Szpilrajn’s theorem shows that transitivity is a sufficient condition for the existence of an ordering extension—that is, an extension of a relation that is reflexive, complete and transitive. The generalization achieved by Suzumura (1976b) represents a major step forward: he shows not only that Suzumura consistency is a weaker sufficient condition for an ordering extension but that it is also necessary and, therefore, provides a clear and unambiguous dividing line between relations that can and that cannot be extended in this fashion. Other commonly-used weakenings of transitivity do not have this property. For example, acyclicity is also necessary for the existence of an ordering extension (this is immediate because the property is implied by Suzumura consistency) but it is not sufficient, and quasi-transitivity is neither necessary nor sufficient for the existence result.

A second important observation is that Suzumura consistency allows for a well-defined closure operation, just as is the case for transitivity. The existence of a closure operation can be useful in various applications. For instance, as will be stated in more detail later on, the analysis of rational choice is simplified considerably if the relation under consideration possesses a well-defined closure. Roughly speaking, observed choice behavior is rationalizable if the choices from feasible sets are in accordance with an underlying goodness relation (a rationalization) that generates these choices as best elements within the respective feasible sets. It is well-known that if a choice function is to be rationalizable, the direct revealed preference relation associated with the observed choices must be respected. That is, if an alternative \(x\) in a feasible set \(S\) is chosen in the presence of each member of \(S\) in some set, then \(x\) must be chosen from \(S\). Further restrictions depend on the additional properties that a rationalizing relation is assumed to possess. For instance, if a rationalization is required to be transitive, it is immediate that it must respect the transitive closure of the direct revealed preference relation. Analogously, if the rationalization is to be Suzumura consistent, there is a parallel restriction—namely, that the Suzumura-consistent closure of the direct revealed preference relation be respected. This allows for the formulation of clear-cut and intuitively plausible necessary and sufficient conditions in the context of Suzumura-consistent rational choice, just as is the
case for transitive rationality. Observe that quasi-transitivity and acyclicity do not have well-defined closure operations because there is no unique way of adding further pairs to a relation in order to arrive at a quasi-transitive or an acyclical relation. As a consequence, necessary and sufficient conditions for quasi-transitive or acyclical rationality are considerably more cumbersome and less intuitive; see, for instance, Bossert and Suzumura (2010, Chapter 4).

The remainder of the paper is structured as follows. The next section provides some basic definitions, followed by a discussion of Suzumura consistency and its most important properties in Section 3. Sections 4, 5 and 6 illustrate how this coherence condition can be applied. In particular, recent applications in the theory of rational choice, choice under uncertainty, and collective decision-making are discussed. Section 7 concludes.

2 Preliminaries

Suppose there is a non-empty universal set of alternatives $X$ the members of which are interpreted as possible outcomes of an individual or a collective decision-making process. A binary relation on $X$ is a set $R \subseteq X \times X$ with the interpretation that, for any two objects $x$ and $y$ in $X$, $(x, y) \in R$ means that $x$ is considered at least as good as $y$ by the decision maker. The asymmetric part of $R$ is given by

$$P(R) = \{(x, y) \mid (x, y) \in R \text{ and } (y, x) \notin R\}$$

and the symmetric part of $R$ is

$$I(R) = \{(x, y) \mid (x, y) \in R \text{ and } (y, x) \in R\}.$$

In keeping with the interpretation of $R$ as a goodness relation, $P(R)$ and $I(R)$ are the better-than relation and the as-good-as relation associated with $R$.

A relation $R$ is reflexive if, for all $x \in X$,

$$(x, x) \in R$$

and $R$ is complete if, for all $x, y \in X$ such that $x \neq y$,

$$(x, y) \in R \text{ or } (y, x) \in R.$$  

The most common coherence property that is imposed on binary relations is transitivity. It requires that if an alternative $x$ is at least as good as an alternative $y$ and $y$, in turn, is at least as good as an alternative $z$, then $x$ must be at least as good as $z$. Thus, $R$ is transitive if, for all $x, y, z \in X$,

$$(x, y) \in R \text{ and } (y, z) \in R \Rightarrow (x, z) \in R.$$  

A reflexive, complete and transitive relation is referred to as an ordering.
The transitive closure $tc(R)$ of a relation $R$ is the smallest transitive relation that contains $R$. Formally, the transitive closure is defined by

$$tc(R) = \{(x, y) \mid \text{there exist } K \in \mathbb{N} \text{ and } x^0, \ldots, x^K \in X \text{ such that } x = x^0, (x^{k-1}, x^k) \in R \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = y\}.$$ 

An important observation is that the transitive closure of a relation is well-defined and unique: for any relation $R$, there exists exactly one transitive closure. In addition, the relation $R$ is itself transitive if and only if it is equal to its transitive closure—that is, if and only if $R = tc(R)$. It can also be verified that, for any two relations $R$ and $R'$, if $R$ is a subset of $R'$, then the transitive closure $tc(R)$ of $R$ must be a subset of the transitive closure $tc(R')$ of $R'$.

Intuitively, the transitive closure of a relation $R$ is obtained by adding every pair of alternatives to $R$ that can be obtained via a chain of pairs in $R$. Moreover, to ensure that the resulting relation is indeed the smallest transitive relation that contains $R$, no further pairs are to be added.

The standard weakenings of transitivity that appear in the requisite literature are quasi-transitivity and acyclicity. Quasi-transitivity requires the asymmetric part of $R$ to be transitive. Thus, $R$ is quasi-transitive if, for all $x, y, z \in X$,

$$(x, y) \in P(R) \text{ and } (y, z) \in P(R) \Rightarrow (x, z) \in P(R).$$

Acyclicity rules out the existence of better-than cycles of any length. That is, $R$ is acyclical if, for all $x, y \in X$,

$$(x, y) \in tc(P(R)) \Rightarrow (y, x) \notin P(R).$$

Clearly, transitivity implies quasi-transitivity and quasi-transitivity implies acyclicity. The reverse implications are not valid.

An operator analogous to the transitive closure does not exist for quasi-transitivity and for acyclicity. This observation is established in the following example.

**Example 1** Let $X = \{x, y, z\}$ and $R = \{(x, x), (x, y), (y, y), (y, z), (z, x), (z, z)\}$. This relation is not acyclical (and, therefore, not quasi-transitive): it follows that $(x, y) \in P(R)$, $(y, z) \in P(R)$ and $(z, x) \in P(R)$. Thus, there is a cycle (which also constitutes a violation of quasi-transitivity) and there is no unique way of obtaining an acyclical relation that contains $R$—note that any one of the pairs $(x, z)$, $(y, x)$ or $(z, y)$ can be added to $R$ in order to break the cycle. Analogously, any one of the two-element sets $\{(z, y), (y, x)\}$, $\{(y, x), (x, z)\}$ or $\{(x, z), (z, y)\}$ can be added to $R$ to obtain a quasi-transitive relation but, again, there is no unique way of doing so. ■

### 3 Suzumura consistency: the fundamentals

Suzumura (1976b) introduces a consistency property that is a weakening of transitivity with intuitive interpretation. Suzumura consistency strengthens acyclicity: while acyclicity rules out cycles all elements of which are instances of the better-than relation,
Suzumura consistency rules out all cycles with at least one instance of betterness. Thus, a relation $R$ that declares $x$ and $y$ to be equally good, $y$ and $z$ to be equally good, and $z$ to be better than $x$ is acyclical—there are no cycles involving betterness only. But $R$ fails to be Suzumura consistent: there is a cycle involving an instance of betterness because $(x, y) \in I(R)$, $(y, z) \in I(R)$ and $(z, x) \in P(R)$ are true—a violation of the property just described. Formally, a relation $R$ is Suzumura consistent if, for all $x, y \in X$,

$$(x, y) \in tc(R) \Rightarrow (y, x) \not\in P(R).$$

That Suzumura consistency implies acyclicity is immediate: note that the consequence $(y, x) \not\in P(R)$ is required whenever $(x, y) \in tc(R)$ in the case of Suzumura consistency, whereas the same consequence only applies under weaker circumstances in the case of acyclicity—namely, only if the chain leading from $x$ to $y$ is composed exclusively of instances of betterness. Transitivity implies Suzumura consistency, and quasi-transitivity and Suzumura consistency are independent. An important special case is obtained when $R$ is reflexive and complete. In the presence of these two properties, transitivity and Suzumura consistency are equivalent and, thus, Suzumura consistency emerges as an intuitively plausible weakening of transitivity. This is not the case for quasi-transitivity (and, thus, acyclicity)—it is straightforward to see that a reflexive, complete and quasi-transitive relation need not be transitive.

To illustrate the intuition underlying Suzumura consistency, consider a situation in which this property is not satisfied—that is, there are alternatives $x_0, \ldots, x_K$ for some $K \in \mathbb{N}$ such that $x_0$ is at least as good as $x^1$ and so on until $x^{K-1}$ is at least as good as $x^K$, and $x^K$ is better than $x_0$. Suppose an agent is in possession of $x^K$. Assuming that all transactions are costless, the agent is willing to exchange $x^K$ for $x^{K-1}$ because the latter is at least as good as the former. This reasoning can be repeated until alternative $x^1$ is reached, which is then exchanged for $x^0$ because $x^0$ is at least as good. Now the agent is in possession of $x^0$ and if she or he is offered $x^K$ in exchange for $x^0$, this is a trade that makes the agent better off. Thus, a chain of weakly favorable trades leads the agent back to where he or she started but there is no sense in which the individual is better off than before these exchanges. Owing to the instance of betterness involved, it may even be the case that the individual is willing to make a payment in addition to relinquishing $x_0$ for $x^K$, in which case there is an instance of a money pump. Avoiding the possibility of such counter-intuitive chains of trades is precisely what Suzumura consistency does.

A consequence of a well-known theorem due to Szpilrajn (1930) is that any transitive relation can be extended to a reflexive, complete and transitive relation. Thus, transitivity is a sufficient condition for the existence of an ordering extension. Suzumura (1976b) succeeds in providing a substantial generalization of this result: he shows that Suzumura consistency is both necessary and sufficient for the existence of such an extension. This is an observation that is of importance not only for applications in economic theory but also in various branches of mathematics.

The notion of an extension is defined as follows. A relation $R'$ on $X$ is an extension of a relation $R$ on $X$ if

$$R \subseteq R' \quad \text{and} \quad P(R) \subseteq P(R').$$
An extension $R'$ has to preserve all comparisons according to the original relation $R$ and, moreover, all original instances of betterness have to be respected. Clearly, the latter requirement is essential in order to avoid degenerate cases in which the universal equal-goodness relation would be an extension of any relation. If $R'$ is an extension of $R$ and $R'$ has the properties of reflexivity, completeness and transitivity, $R'$ is referred to as an ordering extension of $R$.

Szpilrajn (1930) shows that if a relation $R$ on $X$ is reflexive and transitive, then $R$ has an ordering extension. Again, note that the result involves an implication—transitivity is proven to be sufficient for the existence of an ordering extension. Suzumura’s (1976b) extension theorem stated below goes beyond this observation by establishing that Suzumura consistency is both necessary and sufficient for the existence of such an extension.

**Theorem 1** A relation $R$ on $X$ has an ordering extension if and only if $R$ is Suzumura consistent.

I now return to the issue of defining a suitable closure operation. As pointed out in the previous section, it is not possible to obtain a quasi-transitive closure or an acyclical closure of a relation. In contrast, a Suzumura-consistent closure exists for any relation $R$, an observation that can be helpful in many contexts. The Suzumura-consistent closure $sc(R)$ of a relation $R$ on $X$ is defined by letting

$$sc(R) = R \cup \{(x, y) \mid (x, y) \in tc(R) \text{ and } (y, x) \in R\}.$$ 

Clearly, $R$ is a subset of $sc(R)$ which, in turn, is a subset of $tc(R)$. Just as $tc(R)$ is the unique smallest transitive relation containing $R$, $sc(R)$ is the unique smallest Suzumura-consistent relation containing $R$. In addition, a relation $R$ is Suzumura consistent if and only if $R = sc(R)$, and $sc(R)$ is a subset of $sc(R')$ whenever $R$ is a subset of $R'$. The notion of a Suzumura-consistent closure first appears in Bossert, Sprumont and Suzumura (2005).

Here is an example that illustrates how the Suzumura-consistent closure is obtained from a given relation.

**Example 2** Let $X = \{x, y, z, w\}$ and

$$R = \{(x, x), (x, y), (x, w), (y, y), (y, z), (z, x), (z, z), (w, w)\}.$$ 

The corresponding asymmetric and symmetric parts are $P(R) = \{(x, y), (x, w), (y, z), (z, x)\}$ and $I(R) = \{(x, x), (y, y), (z, z), (w, w)\}$. Clearly, the relation $R$ is not Suzumura consistent: observe that $(x, z) \in tc(R)$ and $(z, x) \in P(R)$ and, likewise, $(y, x) \in tc(R)$ and $(x, y) \in P(R)$ as well as $(z, y) \in tc(R)$ and $(y, z) \in P(R)$. According to the definition of $sc(R)$, it is necessary to add

(i) the pair $(x, z)$ as a consequence of $(x, z) \in tc(R)$ and $(z, x) \in R$;
(ii) the pair $(y, x)$ as a consequence of $(y, x) \in tc(R)$ and $(x, y) \in R$;
(iii) the pair $(z, y)$ as a consequence of $(z, y) \in tc(R)$ and $(y, z) \in R$. 


No further pairs are to be added to ensure the minimality property of \( \text{sc}(R) \) and, thus, the Suzumura-consistent closure of \( R \) is given by

\[
\text{tc}(R) = R \cup \{(x, z), (y, x), (z, y)\}
\]

\[
= \{(x, x), (x, y), (x, z), (x, w), (y, x), (y, y), (y, z), (z, x), (z, y), (z, z), (w, w)\}.
\]

The asymmetric and symmetric parts of \( \text{sc}(R) \) are given by

\[
P(\text{sc}(R)) = \{(x, w)\}
\]

and

\[
I(\text{sc}(R)) = \{(x, x), (x, y), (x, z), (y, y), (y, z), (z, x), (z, y), (z, z), (w, w)\}.
\]

The relation \( \text{sc}(R) \) is not transitive: note that, for instance, \((y, x) \in \text{sc}(R) \) and \((x, w) \in \text{sc}(R) \) but \((y, w) \notin \text{sc}(R) \). In this example, therefore, the Suzumura-consistent closure is a strict subset of the transitive closure.

Kotaro Suzumura labels his 1976b contribution “Remarks on the theory of collective choice” but he certainly accomplished more than what this rather modest title may suggest. The applications of Suzumura consistency are by now numerous and go well beyond collective decision-making. To provide a few important examples, the next three sections illustrate how the property can be employed in rational-choice theory, in the theory of choice under uncertainty, and in social-choice theory.

## 4 Rational choice

Consider first the basic decision problem of choosing from a set of feasible options without uncertainty. One of the fundamental questions that goes back as far as Samuelson (1938a,b) is whether observed (or observable) choices can indeed be generated by the process of selecting best elements according to some underlying goodness relation. Samuelson phrases this issue in the context of consumer theory but, clearly, the notion of rational (or rationalizable) choice applies to a more general set of situations. Much of the development of the theory of rational choice on general domains can be attributed to Richter (1966, 1971), Hansson (1968) and Suzumura (1976a, 1977, 1983), to name but a few of the major contributors.

Let \( \mathcal{X} \) denote the power set of \( X \) excluding the empty set, that is, the set of all non-empty subsets of the universal set \( X \). A choice function is a mapping \( C: \Sigma \to \mathcal{X} \) such that \( C(S) \subseteq S \) for all \( S \in \Sigma \), where \( \Sigma \subseteq \mathcal{X} \) with \( \Sigma \neq \emptyset \) is the domain of \( C \). This formulation is completely general in the sense that no structure is imposed on the set \( \Sigma \) of choice situations an agent may be faced with. A prominent example is, of course, the choice problem of a consumer in which \( \Sigma \) is the set of possible budget sets and \( C \) plays the role of a (possibly multi-valued) demand function.

Although originally expressed exclusively in terms of orderings, the notion of rational choice can be defined without any reference to the properties that a rationalizing relation
is assumed to possess. A choice function $C$ is rationalizable if there exists a relation $R$ on $X$ such that
\[ C(S) = \{ x \in S \mid (x, y) \in R \text{ for all } y \in S \} \]
for all $S \in \Sigma$. If such a relation $R$ exists, it is called a rationalization of $C$. In passing, note that this definition refers to best-element rationalizability—the rationalizing relation is required to select all elements that are at least as good as everything that appears in a feasible set. An alternative to this notion is maximal-element rationalizability which demands the existence of a relation such that chosen elements are not dominated by anything in the feasible set—that is, no element of the feasible set is better than a chosen element. However, maximal-element rationalizability is redundant once the properties of the rationalizing relation can be varied: for any definition of best-element rationalizability (where the rationalization may be assumed to possess certain properties), there is an equivalent notion of maximal-element rationalizability (with a rationalization that may possess different properties) but the reverse is not true. See Bossert and Suzumura (2010, Chapter 3) for details.

In formulating the results surveyed in this context, the direct revealed preference relation $R_C \subseteq X \times X$ associated with a choice function $C$ plays an essential role. It is defined as
\[ R_C = \{ (x, y) \mid \text{there exists } S \in \Sigma \text{ such that } x \in C(S) \text{ and } y \in S \}. \]

The importance of this relation is immediate: if there is to be any hope of finding a rationalization $R$ of $C$, this rationalization must respect the direct revealed preference relation $R_C$—that is, if $R$ rationalizes $C$, it must be the case that $R_C \subseteq R$. This observation is due to Samuelson (1938a) and it follows immediately from the requisite definitions. Richter (1971) shows that if no further restrictions are imposed on a rationalization, the following condition is necessary and sufficient for rational choice.

**Direct-revelation coherence** For all $S \in \Sigma$ and for all $x \in S$,
\[ (x, y) \in R_C \text{ for all } y \in S \Rightarrow x \in C(S). \]

This property requires that $R_C$ be respected. If there is an alternative $x$ in the feasible set $S$ such that, for each alternative $y$ in $S$, there is a feasible set $S_y$ that contains $y$ and from which $x$ is chosen, then $x$ must be chosen from $S$ itself. Richter (1971) shows that a choice function $C$ is rationalizable by a relation if and only if $C$ satisfies direct-revelation coherence.

A similar result can be obtained if a rationalizing relation is required to be transitive. In that case, not only $R_C$ needs to be respected but also chains of direct revealed preference. That is, replacing the direct revealed preference relation in the formulation of the above property with its transitive closure yields a necessary and sufficient condition for rationalizability by a transitive relation, a result that also appears in Richter (1971). The requisite property is defined as follows.

**Transitive-closure coherence** For all $S \in \Sigma$ and for all $x \in S$,
\[ (x, y) \in tc(R_C) \text{ for all } y \in S \Rightarrow x \in C(S). \]
Richter (1966, 1971) shows that transitive-closure coherence is necessary and sufficient not only for transitive rationalizability but also for rationalizability by a reflexive, complete and transitive relation. This is a considerably more subtle result than that involving transitivity alone because, in the general case, its proof requires non-constructive methods such as the axiom of choice.

Richter’s (1966, 1971) observations provide clear-cut necessary and sufficient conditions for rationalizability per se and for rationalizability by a transitive (and reflexive and complete) relation. Intuitively, this is the case because there is a transparent principle underlying these notions of rationality: in the former case, the direct revealed preference relation has to be respected; in the latter, the pairs in its transitive closure must be preserved. This observation brings us back to a difficulty that is associated with quasi-transitivity and acyclicity—namely the absence of a well-defined closure operation. Without such a concept, it is impossible to specify precisely what comparisons need to be preserved by a rationalizing relation and, as a result, necessary and sufficient conditions that involve these properties cannot be formulated with this intuitive principle in mind. It is possible to characterize these notions of rationality but the conditions are much more involved and much less transparent; see Bossert and Suzumura (2010, Chapter 4) for an illustration of this point.

In contrast, rationalizability by a Suzumura-consistent relation can be dealt with in a completely analogous fashion: now what needs to be respected is the Suzumura-consistent closure of the direct revealed preference relation. That is, the following condition is necessary and sufficient for this notion of rationality.

**Suzumura-consistent-closure coherence** For all $S \in \Sigma$ and for all $x \in S$,

$$(x, y) \in \text{sc}(R_C) \text{ for all } y \in S \Rightarrow x \in C(S).$$

The requisite characterization result, stated in the following theorem, is due to Bossert, Sprumont and Suzumura (2005).

**Theorem 2** A choice function $C$ is rationalizable by a Suzumura-consistent relation if and only if $C$ satisfies Suzumura-consistent-closure coherence.

I conclude this section with two examples that illustrate the rationalizability criterion established in the above theorem.

**Example 3** Let $X = \{x, y, z, w\}$, $\Sigma = \{\{x, y\}, \{x, z\}, \{x, z, w\}, \{y, z\}\}$ and define the choice function $C$ by

$$C(\{x, y\}) = \{x\}, C(\{x, z\}) = \{x, z\}, C(\{x, z, w\}) = \{x, w\}, C(\{y, z\}) = \{y\}.$$ 

The direct revealed preference relation $R_C$ that corresponds to $C$ is

$$R_C = \{(x, x), (x, y), (x, z), (x, w), (y, y), (y, z), (z, x), (z, z), (w, x), (w, z), (w, w)\}.$$ 

This choice function is not rationalizable by a Suzumura-consistent relation. Note that Suzumura-consistent-closure coherence is violated because $(z, x) \in \text{sc}(R_C)$, $(z, z) \in \text{sc}(R_C)$.
and \((z, w) \in sc(R_C)\) but \(z \not\in C(\{x, z, w\})\). However, the choice function is rationalizable by the relation \(R_C\), which fails to be Suzumura consistent because \((z, w) \in tc(R_C)\) and \((w, z) \in P(R_C)\).

**Example 4** Now suppose that \(X = \{x, y, z, w\}\), \(\Sigma = \{\{x, y\}, \{x, z\}, \{x, z, w\}, \{y, z\}\}\) and define the choice function \(C\) by

\[
C(\{x, y\}) = \{x, y\}, C(\{x, z\}) = \{x, z\}, C(\{x, z, w\}) = \{x\}, C(\{y, z\}) = \{y, z\}.
\]

As is straightforward to verify, this choice function is rationalizable by the Suzumura-consistent relation \(R\) defined by

\[
R = \{(x, x), (x, y), (x, z), (x, w), (y, x), (y, y), (y, z), (z, x), (z, y), (z, z)\} = R_C = sc(R_C).
\]

However, \(C\) cannot be rationalized by a transitive relation. Transitive-closure coherence is violated because \((z, x) \in tc(R_C)\), \((z, z) \in tc(R_C)\) and \((z, w) \in tc(R_C)\) but \(z \not\in C(\{x, z, w\})\).

**5 Choice under uncertainty**

The expected-utility criterion has a rich history in the analysis of decision-making under uncertainty, starting with the seminal contribution of von Neumann and Morgenstern (1944; 1947). In spite of some criticisms regarding its descriptive power, expected-utility theory continues to be based on a solid normative foundation and, therefore, it seems premature to reject it altogether. There are numerous approaches in the literature that attempt to resolve perceived paradoxes such as those alluded to by Allais (1953), Kahneman and Tversky (1979) and Machina (1983). But there is an alternative to abandoning the notion of expected utility altogether. The so-called paradoxes are frequently explained by appealing to the idea that many choice situations are of a complex nature and, therefore, the agent may not be able to apply the expected-utility criterion. By the same token, it can be argued that some comparisons of probability distributions may be too complex and the agent encounters situations that involve non-comparable options. Thus, dropping completeness as a requirement imposed on a decision rule would seem to be in line with this interpretation. Of course, if one continues to require transitivity, this is not of much help—as is well-known, completeness is implied if transitivity is required in addition to the standard expected-utility axioms. Therefore, a more promising proposal consists of eliminating completeness from the list of requirements and weakening transitivity to Suzumura consistency. The result is a class of decision rules that generalize the expected-utility criterion and, because of the possibility of non-comparability, suitably selected members of this class manage to avoid essentially all reversals.

For the purposes of this section, the set of alternatives \(X = \{x_1, \ldots, x_n\}\) is assumed to be finite, where \(n \in \mathbb{N} \setminus \{1, 2\}\). The one-alternative and two-alternative cases are excluded because they are trivial. It is assumed that there exist two distinct alternatives \(x_j\) and \(x_k\) in \(X\) such that the probability distribution that assigns a probability of one to \(x_j\) is better
than the distribution that yields \( x_k \) with certainty. Without loss of generality, suppose that these alternatives are given by \( x_j = x_1 \) and \( x_k = x_n \). The unit simplex in \( \mathbb{R}_+^n \) is denoted by \( \Delta = \{ p \in \mathbb{R}_+^n | \sum_{i=1}^n p_i = 1 \} \), and its members are the possible probability distributions on \( X \). Furthermore, the relation \( R \subseteq \Delta \times \Delta \) is now defined on the set \( \Delta \) of all probability distributions and it is interpreted as a decision rule under uncertainty.

The following three properties are standard in decision theory: the first is a continuity condition, the second is a monotonicity requirement, and the third amounts to separability.

**Solvability** For all \( p \in \Delta \), there exists \( \alpha \in [0,1] \) such that

\[
\left( p, (\alpha, 0, \ldots, 0, 1 - \alpha) \right) \in I(R).
\]

**Monotonicity** For all \( \alpha, \beta \in [0,1] \),

\[
\left( (\alpha, 0, \ldots, 0, 1 - \alpha), (\beta, 0, \ldots, 0, 1 - \beta) \right) \in R \iff \alpha \geq \beta.
\]

**Independence** For all \( p, q \in \Delta \) and for all \( \alpha, \beta, \gamma \in [0,1] \), if

\[
\left( p, (\alpha, 0, \ldots, 0, 1 - \alpha) \right) \in I(R) \quad \text{and} \quad \left( q, (\beta, 0, \ldots, 0, 1 - \beta) \right) \in I(R),
\]

then

\[
\left( \gamma p + (1 - \gamma)q, \gamma(\alpha, 0, \ldots, 0, 1 - \alpha) + (1 - \gamma)(\beta, 0, \ldots, 0, 1 - \beta) \right) \in I(R).
\]

The characterization of the generalized expected-utility criterion is presented in the following theorem, the proof of which can be found in Bossert and Suzumura (2015). See also Kreps (1988).

**Theorem 3** Suppose that \( X \) contains at least three alternatives and that \( R \) is a relation on \( \Delta \) such that \( \left( (1, 0, \ldots, 0), (0, \ldots, 0, 1) \right) \in P(R) \). The relation \( R \) satisfies Suzumura consistency, solvability, monotonicity and independence if and only if there exists a function \( U : X \to \mathbb{R} \) such that the pair \((R, U)\) satisfies

1. \( U(x_1) = 1 \) and \( U(x_n) = 0 \);
2. \( \left( (\alpha, 0, \ldots, 0, 1 - \alpha), (\beta, 0, \ldots, 0, 1 - \beta) \right) \in R \iff \alpha \geq \beta \) for all \( \alpha, \beta \in [0,1] \);
3. \( \left( p, \left( \sum_{i=1}^n p_i U(x_i), 0, \ldots, 0, 1 - \sum_{i=1}^n p_i U(x_i) \right) \right) \in I(R) \) for all \( p \in \Delta \);
4. \( (p, q) \in I(R) \Rightarrow \sum_{i=1}^n p_i U(x_i) = \sum_{i=1}^n q_i U(x_i) \) for all \( p, q \in \Delta \);
5. \( (p, q) \in P(R) \Rightarrow \sum_{i=1}^n p_i U(x_i) > \sum_{i=1}^n q_i U(x_i) \) for all \( p, q \in \Delta \).
The classical expected-utility criterion is a special case of the class characterized in Theorem 3. Another special case is that corresponding to the criterion \( R \) defined by

\[
\left( (\alpha, 0, \ldots, 0, 1 - \alpha), (\beta, 0, \ldots, 0, 1 - \beta) \right) \in R \iff \alpha \geq \beta
\]

for all \( \alpha, \beta \in [0, 1] \) and

\[
\left( p, \left( \sum_{i=1}^{n} p_i U(x_i), 0, \ldots, 0, 1 - \sum_{i=1}^{n} p_i U(x_i) \right) \right) \in I(R)
\]

for all \( p \in \Delta \), where \( U(x_1) = 1 \) and \( U(x_n) = 0 \). The expected-utility criterion itself ensures maximal possible comparability within the class (because the criterion is complete), whereas the example just defined is minimal in the sense that no pairs are added to \( R \) that are not immediately imposed by some of the characterizing properties. Note that this class is very rich: any criterion that is between these two extremes belongs to it.

As mentioned earlier in this section, suitable choices from the class of decision rules characterized in Theorem 3 rule out essentially all possibly problematic reversals. The following two examples are taken from Bossert and Suzumura (2015).

**Example 5** This example is a special case of the common-consequence effect (see Machina, 1983), known as the Allais (1953) paradox. The set of alternatives is \( X = \{5, 1, 0\} \), where \( x_1 = 5 \) stands for receiving five million dollars, \( x_2 = 1 \) represents one million dollars and \( x_3 = 0 \) is an alternative in which the amount paid to the agent is zero. Experimental evidence such as that presented by Kahneman and Tversky (1979) seems to suggest that the following rankings are frequently observed. They involve the pair of distributions \( p = (0, 1, 0) \) versus \( q = (0.1, 0.89, 0.01) \) and the pair \( p' = (0, 0.11, 0.89) \) versus \( q' = (0.1, 0, 0.9) \). Numerous experiments reveal that agents appear to rank \( p \) as better than \( q \) and \( q' \) as better than \( p' \). These two rankings together are not consistent with classical expected utility theory. For any function \( U: X \rightarrow \mathbb{R} \) such that \( U(5) = 1 \) and \( U(0) = 0 \), \( (p, q) \in P(R) \) implies \( U(1) > 0.1 U(5) + 0.89 U(1) + 0.01 U(0) \) and hence \( U(1) > 1/11 \). But \( (q', p') \in P(R) \) yields \( 0.1 U(5) + 0.9 U(0) > 0.11 U(1) + 0.89 U(0) \) which immediately implies \( U(1) < 1/11 \), a contradiction. It is, however, no problem to reconcile these rankings by means of a generalized expected-utility criterion. Choose a function \( U \) such that \( U(1) > 1/11 \) and a generalized criterion \( R \) from the class characterized in Theorem 3 such that \( (p, q) \in P(R) \) and \( p' \) and \( q' \) are non-comparable. Alternatively, it is possible to select \( V \) such that \( V(1) < 1/11 \), \( (q', p') \in P(R) \) and \( p \) and \( q \) are non-comparable. Each of these options avoids the paradox. \( \blacksquare \)

**Example 6** Now consider the certainty effect (Kahneman and Tversky, 1979) which is a specific instance of the common-ratio effect (again, see Machina, 1983). Let \( X = \{6, 3, 0\} \), where \( x_1 = 6 \) stands for receiving \$6,000, \( x_2 = 3 \) means that the agent receives \$3,000, and if \( x_3 = 0 \), the agent gets nothing. Again, there are two pairs of distributions to be considered—this time, they are \( p = (0, 0.9, 0.1) \) versus \( q = (0.45, 0, 0.55) \) and \( p' = (0, 0.002, 0.998) \) versus \( q' = (0.001, 0, 0.999) \). A common observation from experimental
studies appears to be that \( p \) is considered better than \( q \) and \( q' \) is better than \( p' \). As is the case for the Allais paradox, these two rankings taken together cannot be obtained in accordance with the classical expected-utility criterion. For any function \( U: X \to \mathbb{R} \) such that \( U(6) = 1 \) and \( U(0) = 0 \), \((p, q) \in P(R)\) means that \( 0.9U(3) + 0.1U(0) > 0.45U(6) + 0.55U(0) \) and, therefore, \( U(3) > 1/2 \). However, \((q', p') \in P(R)\) implies \( 0.001U(6) + 0.999U(0) > 0.002U(3) + 0.998U(0) \) and hence \( U(3) < 1/2 \), which is a contradiction. Again, this perceived paradox can be resolved by means of a suitably chosen generalized expected-utility criterion as in the previous example. ■

6 Collective choice

This section illustrates how Suzumura consistency can be employed in the context of Arrovian (1951; 1963) social choice. Suppose that the universal set \( X \) contains at least three alternatives, that is, the cardinality \(|X|\) of \( X \) is greater than or equal to three. The set of all reflexive and Suzumura-consistent relations on \( X \) is denoted by \( \mathcal{C} \), and \( \mathcal{R} \) is the set of all orderings. The set of agents in a society is \( N \), where \(|N|\) is finite and greater than or equal to two. A profile is an \(|N|\)-tuple \( R = (R_1, \ldots, R_{|N|}) \in \mathcal{R}^{[N]} \), where, for all \( i \in N \), \( R_i \) is individual \( i \)'s goodness relation.

A Suzumura-consistent collective-choice rule is a mapping \( f: \mathcal{D} \to \mathcal{C} \) where \( \mathcal{D} \subseteq \mathcal{R}^{[N]} \) is the domain of \( f \). Note that individual goodness relations are assumed to be orderings but collective relations need not be complete and transitive—but they are assumed to be reflexive and Suzumura consistent. The following properties are standard in social-choice theory.

**Unrestricted domain** \( \mathcal{D} = \mathcal{R}^{[N]} \).

**Strong Pareto** For all \( x, y \in X \) and for all \( R \in \mathcal{D} \),

(i) if \((x, y) \in R_i \) for all \( i \in N \), then \((x, y) \in f(R)\);

(ii) if \((x, y) \in R_i \) for all \( i \in N \) and there exists \( j \in N \) such that \((x, y) \in P(R_j) \), then \((x, y) \in P(f(R))\).

**Neutrality** For all \( x, y, z, w \in X \) and for all \( R, R' \in \mathcal{D} \), if

\[(x, y) \in R_i \iff (z, w) \in R'_i \text{ for all } i \in N \text{ and } (y, x) \in R_i \iff (w, z) \in R'_i \text{ for all } i \in N,\]

then \((x, y) \in f(R) \iff (z, w) \in f(R')\) and \((y, x) \in f(R) \iff (w, z) \in f(R')\).

**Anonymity** For all bijections \( \rho: N \to N \) and for all \( R, R' \in \mathcal{D} \),

\[R_i = R'_{\rho(i)} \text{ for all } i \in N \implies f(R) = f(R').\]

Arrow’s (1951; 1963) impossibility theorem states that if a collective-choice rule satisfies unrestricted domain, weak Pareto (a weakening of the above Pareto condition), independence of irrelevant alternatives (a weakening of neutrality) and is required to generate collective orderings, then there must be a dictator—an individual whose betterness relations...
are always respected in any profile. (The anonymity property is a strengthening of the requirement that there be no dictator.) If collective goodness relations are quasi-transitive and complete, the Pareto extension rule can be characterized by means of unrestricted domain, strong Pareto, independence of irrelevant alternatives and anonymity; see Sen (1970, Theorem 5*3). Weymark (1984, Theorem 3) allows social goodness relations to be incomplete but imposes full transitivity and obtains a characterization of the Pareto rule using the axioms of Sen’s (1970, Theorem 5*3) result. The Pareto rule, also referred to as the unanimity rule, declares an alternative to be collectively at least as good as another if and only if every individual considers the former to be at least as good as the latter. Clearly, the social relations thus defined are not necessarily orderings—as soon as at least two agents express different betterness rankings between two alternatives, the options are non-comparable according to this rule. The Pareto extension rule is obtained by replacing non-comparability with equal goodness, thereby generating a complete relation which need no longer be transitive. However, the resulting collective goodness relation is quasi-transitive. Formally, the Pareto rule $f^p$ is defined by letting, for all $x, y \in X$ and for all $R \in R^{|N|}$,

$$(x, y) \in f^p(R) \iff (x, y) \in R_i \text{ for all } i \in N,$$

and the Pareto extension rule $f^e$ is defined by

$$(x, y) \in f^e(R) \iff (y, x) \notin P(f^p(R))$$

for all $x, y \in X$ and for all $R \in R^{|N|}$.

To define the class of Suzumura-consistent collective-choice rules that satisfy unrestricted domain, strong Pareto, neutrality and anonymity, some further notation is required. Let $B(x, y; R)$ denote the set of individuals such that $x \in X$ is better than $y \in X$ in the profile $R \in R^{|N|}$. That is, for all $x, y \in X$ and for all $R \in R^{|N|}$,

$$B(x, y; R) = \{i \in N \mid (x, y) \in P(R_i)\}.$$  

Now let

$$S = \{(w, \ell) \in \{0, \ldots, |N|\}^2 \mid 0 \leq |X| \ell < w + \ell \leq |N|\} \cup \{(0, 0)\}$$

and define

$$\Omega = \{S \subseteq S \mid (w, 0) \in S \text{ for all } w \in \{0, \ldots, |N|\}\}.$$  

For $S \in \Omega$, define the $S$-rule $f^S: R^{|N|} \rightarrow C$ by

$$(x, y) \in f^S(R) \iff \text{there exists } (w, \ell) \in S \text{ such that } |B(x, y; R)| = w \text{ and } |B(y, x; R)| = \ell$$

for all $x, y \in X$ and for all $R \in R^{|N|}$. The set $S$ specifies the pairs of numbers of agents who have to consider an alternative $x$ better (respectively worse) than an alternative $y$ in order to declare $x$ to be socially at least as good as $y$ according to the profile under consideration. Unrestricted domain is satisfied because each $S$-rule is defined for all
possible profiles. Strong Pareto follows from the requirement that the pairs \((w, 0)\) be in \(S\) in the definition of \(\Omega\). Clearly, neutrality is satisfied because the parameters do not depend on the labels attached to alternatives to be ranked. Analogously, because only the number of individuals matters and not their identities, each \(S\)-rule is anonymous. Reflexivity of the social relation follows from the reflexivity of the individual goodness relations and the observation that \((0, 0) \in S\) for all \(S \in \Omega\). The collective goodness relation \(R^S\) is Suzumura consistent as a consequence of the restrictions imposed on the pairs \((w, \ell)\) in the definition of \(S\). Two examples are used to illustrate these rules.

**Example 7** The Pareto rule is the special case that is obtained for

\[ S = \{(w, 0) \mid w \in \{0, \ldots, |N|\}\}. \]

If \(|X| \geq |N|\), this is the only \(S\)-rule. This is the case because only pairs \((w, \ell)\) where \(\ell = 0\) are in \(S\) in the presence of this inequality. To see this, suppose, to the contrary, that there exists \((w, \ell) \in S\) such that \(\ell > 0\). Because \((w, \ell) \in S\), it follows that \(|N| \geq w + \ell > |X|\ell > 0\). Combined with \(|X| \geq |N|\), this implies \(|N| > |N|\ell\) which is impossible if \(\ell > 0\).

This example shows that if \(|X| \geq |N|\), the characterization of the class of \(S\)-rules presented below provides an alternative characterization of the Pareto rule. The axiomatization differs from Weymark’s (1984) in that independence of irrelevant alternatives is strengthened to neutrality and transitivity is weakened to Suzumura consistency. Also, in view of the above example, it follows that if \(|X| \geq |N|\), transitivity is implied by the conjunction of Suzumura consistency and the other axioms employed in the requisite theorem. However, if \(|X| < |N|\), the Pareto rule is not the only \(S\)-rule, as demonstrated by the following example.

**Example 8** Suppose that \(|X| = 3\) and \(|N| = 7\), and consider the collective-choice rule \(f^S\) corresponding to the set

\[ S = \{(w, 0) \mid w \in \{0, \ldots, |N|\}\} \cup \{(6, 1), (5, 1), (5, 2)\}. \]

For \((w, \ell) = (6, 1)\), it follows that \(w + \ell = 6 + 1 = 7 > 3 \cdot 1 = |X|\ell\). Analogously, substituting \((w, \ell) = (5, 1)\) yields \(w + \ell = 5 + 1 = 6 > 3 \cdot 1 = |X|\ell\) and, finally, for \((w, \ell) = (5, 2)\), it follows that \(w + \ell = 5 + 2 = 7 > 3 \cdot 2 = |X|\ell\). Thus, the required inequalities in the definition of \(S\) are satisfied and the rule is well-defined (and, thus, produces Suzumura-consistent social goodness relations).

It may be of interest to note that, in the above example, an alternative \(x\) can be declared socially better than an alternative \(y\) as soon as at least five individuals out of seven consider \(x\) better than \(y\) (and at most two individuals consider \(y\) better than \(x\)). Thus, full unanimity is not required—as long as roughly 71% of the individuals are favorably disposed towards an alternative \(x\) over an alternative \(y\), the rest of the population cannot prevent \(x\) from being declared socially better than \(y\). Thus, the example illustrates that a considerably richer class of possible collective-choice rules results if social goodness relations are required to be Suzumura consistent instead of assuming quasi-transitivity together with completeness or transitivity.
The $S$-rules are the only Suzumura-consistent collective-choice rules that satisfy unrestricted domain, strong Pareto, neutrality and anonymity. This characterization theorem is the main result of Bossert and Suzumura (2008).

**Theorem 4** A Suzumura-consistent collective-choice rule $f$ satisfies unrestricted domain, strong Pareto, neutrality and anonymity if and only if there exists $S \in \Omega$ such that $f = f^S$.

Note that, unlike the results obtained by Sen (1970, Theorem 5*3) and by Weymark (1984, Theorem 3), Theorem 4 employs neutrality rather than the weaker property of independence of irrelevant alternatives. The collective-choice rules characterized in the two earlier contributions also satisfy neutrality, which is implied once the conditions imposed there (notably quasi-transitivity together with completeness or transitivity) are added to the independence axiom. Because Suzumura consistency is not sufficient for this implication, the stronger property has to be assumed explicitly in Bossert and Suzumura’s (2008) characterization.

### 7 Concluding remarks

This paper provides an appreciation of Suzumura consistency, along with a few examples to illustrate its use in various branches of economic theory. The overview presented here is by no means exhaustive; other recent examples include non-probabilistic models of choice under uncertainty (Bossert and Suzumura, 2012a) and social choice with possibly infinite populations (Bossert and Suzumura, 2012b). A comprehensive review of some earlier contributions can be found in Bossert and Suzumura (2010).

I conclude with expressing the hope that this brief survey serves as a motivation for further work. In particular, it may be useful to explore settings in which the application of coherence properties that are weaker than full transitivity holds some promise when it comes to the task of obtaining richer possibility results than those that are presently available.

### References


