Vote budgets and Dodgson’s method of marks
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\textbf{Abstract.} We examine voting rules that are inspired by Dodgson’s method of marks (to be distinguished from the procedure that is commonly referred to as Dodgson’s rule) by means of two criteria. Each voter decides how to allocate a vote budget (which is common to all voters, and need not be exhausted) to the candidates. Our first criterion is a richness condition: we demand that, for any possible preference ordering a voter may have, there is a feasible allocation of votes that reflects these preferences. A (tight) lower bound on the vote budget is established. Adding a strategy-proofness condition as a second criterion, we recommend that the vote budget be given by the lower bound determined in our first result. \textit{Journal of Economic Literature} Classification Nos.: D71, D72.
1 Introduction

Charles Lutwidge Dodgson, alias Lewis Carroll, occupies a unique niche in the historical evolution of social choice theory with two distinct contributions that fill in the gap between the pioneering initiatives by Jean-Charles de Borda and Marquis de Condorcet in the 18th century, on the one hand, and the seminal work of Kenneth Arrow and Duncan Black in the 1940s and 1950s, on the other. Dodgson’s work appeared in his pamphlets Dodgson (1873) and Dodgson (1876), both of which are reprinted in McLean and Urken (1995). Originally, these two essays were circulated only among a few Oxford dons until Black rediscovered and reproduced them in Black (1958; reprinted in McLean et al., 1998). The first, and better known, contribution is on the voting procedure that is widely referred to as Dodgson’s Rule, according to which all voters submit ordered lists of candidates in accordance with their preferences, and the winner is the candidate for whom we must perform the minimum number of pairwise preference swaps between candidates before he or she becomes a Condorcet winner. This rule is suggested in Dodgson (1876, Section 3). The second—almost parenthetical—contribution is his Method of Marks, according to which ‘a certain number of marks is fixed, which each elector shall have at his disposal; he may assign them all to one candidate, or divide them among several candidates, in proportion to their eligibility; and the candidate who gets the greatest total of marks is the winner’ (Dodgson, 1873, Chapter 1, Section 5). Feldman (1979) also mentions Dodgson’s method of marks in his examination of classical voting procedures in the context of the strategic misrepresentation of voters’ preferences. It is worth noting here that Dodgson made a third contribution to the theory of voting and elections, which is on the topic of proportional representation; see McLean et al. (1996) for details on this latter part of Dodgson’s work.

This paper is devoted to a re-examination of Dodgson’s method of marks in the context of assessing the merits of possible voting procedures. Our intention is to incorporate this proposal authored by one of the most eminent historical figures in voting theory into the current discourse that involves the foundations of actual election methods. We hope to thereby provide a contribution that is of sound theoretical substance and, at the same time, has direct applications in the political arena. The practical relevance of possible electoral reforms is evident—there is a continuing debate that focuses on the suitability of existing electoral systems in many jurisdictions. For instance, the United States Electoral College comes under renewed scrutiny essentially every time a President is elected who fails to obtain the majority of the popular vote. Likewise, the first-past-the-post system employed at the federal level in Canada (and elsewhere) has been the subject of well-founded criticism for decades because of its often blatant disregard of the popular vote and, although the Canadian government elected in 2015 made electoral reform an essential piece of their platform, it abandoned the project soon after coming into power. These are but a couple of examples to illustrate that the design of alternative voting systems is a topic that continues to be of great practical importance in democracies throughout the world.

The formal analysis of voting rules goes back at least to the late eighteenth century. In particular, the seminal contributions of Borda (1781) and Condorcet (1785) continue to this day to have a significant impact on debates that concern electoral procedures. Their contemporaries Morales (1797) and Daunou (1803) provide further discussions of Borda’s and
Condorcet’s original proposals. Morales emerges as a supporter of Borda’s view, whereas Daunou is more favorably disposed towards Condorcet. Although appearing in print only in 1781, some of Borda’s ideas were already presented as early as 1770; see McLean and Urken (1995, p. 83). Translations of these four fundamental essays are, among others, reprinted in McLean and Urken’s (1995) collection of classics of social choice. Borda’s (1781) rule is also discussed by Black (1958, 1976), and Fishburn and Gehrlein (1976) illustrate some features shared by the approaches of Condorcet and Borda. Characterizations of the Borda rule appear in Young (1974) and in Nitzan and Rubinstein (1981). Arguments in favor of the Borda rule, particularly in the context of its relative susceptibility to manipulation, can be found in Saari (1990).

About a century after Borda initially formulated his views on elections by ballot, Dodgson (1873) defined his method of marks as outlined above. It is important to note that the method of marks differs from what is often referred to as Dodgson’s rule or Dodgson’s procedure (see Dodgson, 1876), which is another (Condorcet-consistent) voting rule. There is also a link with the so-called point-distribution schemes or cumulative-voting procedures discussed in Brams and Fishburn (2002, Section 10). The latter two types of rules are typically associated with elections in which \( k \geq 2 \) candidates are to be chosen; these choose-\( k \) elections are not the focus of the approach followed here, however. We note that variants of cumulative voting are quite commonly used, especially at local levels. For instance, in their own words, Pildes and Donoghue (1995, p. 242) ‘. . . provide detailed quantitative and qualitative information, from one of the longest running experiments with cumulative voting in the United States, on the experience of cumulative voting in actual practice.’

There is an important difference between positional voting procedures such as the Borda count or the plurality voting rule, on the one hand, and the method of marks, on the other. As opposed to positional voting rules, the assignment of weights to candidates in accordance with a voter’s preferences is not imposed by the rule designer in Dodgson’s method of marks. Instead, the allocation of marks is left to the voter subject to the vote budget that is common to all electors. This feature is also present (to a lesser extent) in approval voting (Brams and Fishburn, 1978), where a voter is permitted to allocate a total of \( m - 1 \) votes to \( m \) candidates with the additional restriction that each candidate may receive at most one vote. Flexible scoring rules (of which approval voting is an example) are discussed by Baharad and Nitzan (2002, 2016). See, for instance, Fishburn (1973), Gärdenfors (1973), Young (1975) and Pattanaik (2002) for more on positional voting rules.

The class of voting rules examined here is inspired by Dodgson’s method of marks. As outlined by Dodgson (1873), a vote budget is given to each voter and the individual elector gets to decide how to allocate her or his budget to the candidates. Dodgson (1873) himself did not address the issue of how many votes are to be given to an elector. However, this question seems to be of crucial importance when assessing the merits of his method of marks and, consequently, our goal is to provide an answer in this paper. Primarily, we focus on the relative flexibility of such rules when it comes to their ability to accommodate the expression of any possible underlying preference ordering that a voter may have. We phrase the relevant requirement as a richness condition: we demand that, for any possible preference ordering a voter may have, there is a feasible allocation of votes that reflects these preferences. Our first observation establishes a (tight) lower bound on the number
of votes available to an elector that guarantees this richness property to be satisfied. The richness question is also raised in the context of point-voting schemes by Nitzan et al. (1980). However, their formulation allows the vote budget to be perfectly divisible and, as a consequence, it follows immediately that their point-voting method permits any possible preference ordering to be representable by a suitably chosen feasible ballot. This contrasts with our (and Dodgson’s) setting so that their results do not apply here. While allowing for non-integer votes (such as fractions) may simplify the theoretical analysis of voting rules, we think that the integer requirement is essential for voting rules that are to be used in practice; we are not aware of any actual method that allows voters to cast ballots that do not respect the integer constraint. This is particularly relevant here because the votes are assigned by the electors rather than by the designer of the rule.

A second criterion that can be consulted when assessing the relative merits of voting rules is their susceptibility to the strategic misrepresentation of individual preferences. The fundamental result in this context is the well-known impossibility theorem established by Gibbard (1973) and Satterthwaite (1975), proving that any single-valued social choice function (or resolute social choice function in the terminology of Gärfenfors, 1976, 1977) the range of which contains at least three outcomes must be dictatorial or manipulable by strategic misrepresentation. Various proofs of this result are now available and several among them make use of Arrow’s (1951; 1963; 2012) seminal theorem. See Barberà (2011) for an extensive survey of strategy-proofness in the theory of social choice. The Gibbard-Satterthwaite theorem does not directly apply to positional voting rules such as the Borda count or plurality voting because these procedures do not generate single-valued choices in all situations. However, there are contributions in the literature that nevertheless illustrate the vulnerability of such methods to strategic misrepresentation. Some of these deal explicitly with (multi-valued) social choice correspondences, including those of Gärfenfors (1976) and Kelly (1977). The definition of strategy-proofness is adapted to the multiple-outcome setting by supplementing individual preferences defined on alternatives with assumptions regarding the plausible ranking of (some) sets of outcomes. Barberà et al. (2001) adopt an expected-utility framework.

There are a few contributions that deal with the strategic misrepresentation of individual preferences specifically in the context of positional rules and related voting schemes. For instance, Nitzan et al. (1980) study a variation of voting by ballot with vote budgets where the budget is normalized to one and votes need not be integers, thereby deviating from the interpretation underlying Dodgson’s (1873)—and our—approach. Moreover, they assume that the entire budget is to be allocated. Similar remarks apply to Nitzan (1985). Chamberlin’s (1985) comparative examination of four voting methods (including the Borda rule and plurality voting) is different in nature because it is based on a mathematical-programming approach with the use of data that are generated by means of a Monte Carlo process. Saari (1990) bases his analysis of the relative strategy-proofness of positional voting rules on distributional assumptions, and Kelly (1993) employs sampling experiments to address manipulability issues. A more recent approach by Favardin et al. (2002) utilizes the Borda rule and the Copeland (1951) method as representatives of positional and Condorcet-based rules, respectively, to conclude that the Copeland method fares better than the Borda count when it comes to strategy-proofness considerations. A feature that is common to
some of these contributions (such as Nitzan, 1985, Kelly, 1993, and Favardin et al., 2002) is that they employ tie-breaking rules to force single-valued social choices.

We use a natural notion of strategy-proofness that is sufficient to conclude that the lower bound for a vote budget established in our richness result is itself the best choice rather than any budget that exceeds this bound. Our recommendation thus consists of a specific proposal for a vote budget that depends on the number of candidates available in an election.

Ours is not the first contribution that analyses point-distribution procedures and related voting rules. However, the question of choosing an optimal value of the vote budget seems to have been left completely open so far, and the present paper is intended to fill this gap.

Section 2 introduces the basic definitions used in this paper. In Sections 3 and 4, the criteria of richness and manipulability are analysed. Section 5 concludes with a discussion of related voting rules.

2 Definitions

We use $\mathbb{N}$ (resp. $\mathbb{N}_0$) to denote the set of positive (resp. non-negative) integers. The set of all (resp. all positive) real numbers is $\mathbb{R}$ (resp. $\mathbb{R}_+^+$). The finite set of individuals (or voters, or electors) is $N = \{1, \ldots, n\}$ and the finite set of alternatives (or candidates) is $X = \{x_1, \ldots, x_m\}$. We assume that there are at least two voters and at least two candidates, that is, $n, m \in \mathbb{N} \setminus \{1\}$. The set of all non-empty subsets of $X$ is $X^*$.

Each voter $i \in N$ has a preference ordering $R_i$ on $X$ (with the associated strict preference relation $P_i$ and the associated indifference relation $I_i$), and the set of all possible orderings on $X$ is denoted by $\mathcal{R}$. As is common practice in voting theory, we use binary relations interpreted as individual preferences as the basic information regarding a voter’s assessment of the candidates.

A preference profile is an $n$-tuple $\mathbf{R} = (R_1, \ldots, R_n) \in \mathcal{R}^n$. Furthermore, each elector $i \in N$ has a vote budget $v \in \mathbb{N}$ to be allocated among the candidates. Voting proceeds by ballots and, given the vote budget $v$, the set of possible ballots for voter $i \in N$ is

$$B(v) = \left\{ b_i = (b_i(x_1), \ldots, b_i(x_m)) \in \mathbb{N}_0^m \left| \sum_{k=1}^m b_i(x_k) \leq v \right. \right\}$$

where, for each $k \in \{1, \ldots, m\}$, $b_i(x_k)$ is the number of votes assigned to candidate $x_k$ by voter $i$. The vote budget is the same for everyone to ensure that all voters are treated anonymously, with no attention paid to their identities. This budget need not be exhausted—for example, a ballot with no votes assigned to any of the candidates can be used to express universal indifference. The possibility of not using up the entirety of one’s vote budget is essential if no possible preference ordering is to be excluded. If, for instance, there are five candidates and a voter has a budget of twelve votes available, any exhaustive assignment of votes must be such that some candidates receive more votes than others so that expressing universal indifference is not an available option.

Interestingly, even though this is not explicitly stated, it seems to us that this feature of allowing non-exhaustive ballots is what Dodgson (1873; 1995, p. 283) intended when
defining his method of marks. The passage ‘...or divide them among several candidates, in proportion to their eligibility...’ suggests to us that he must have had in mind a voting rule that allows for any possible preference ordering of an elector; the phrase ‘in proportion to their eligibility’ seems to refer to nothing other than the relative rankings of the candidates according to the voter’s preferences. This interpretation leads us directly to the notion of richness, the central property analysed in this paper.

A profile of ballots is an $n$-tuple $b = (b_1, \ldots, b_n) \in B(v)^n$. For each vote budget $v \in \mathbb{N}$, there is an associated voting rule $C_v: B(v)^n \rightarrow \mathcal{X}$ that assigns a set of collectively chosen candidates to each possible profile. The correspondence $C_v$ is defined by letting, for all $b \in B(v)^n$,

$$C_v(b) = \left\{ x \in X \left| \sum_{i=1}^{n} b_i(x) \geq \sum_{i=1}^{n} b_i(y) \forall y \in X \right. \right\}.$$

As mentioned in the introduction, this class of voting rules bears a family resemblance to the point-distribution procedures (or cumulative-voting procedures) discussed in Brams and Fishburn (2002, Section 10). However, their focus is on elections in which $k \geq 2$ candidates are to be chosen, whereas our approach does not encompass this specific interpretation. A ballot $b_i \in B(v)$ is sincere for an ordering $R_i \in \mathcal{R}$ if and only if, for all $x, y \in X$,

$$b_i(x) \geq b_i(y) \Leftrightarrow x R_i y.$$

A ballot $b_i \in B(v)$ is insincere for an ordering $R_i \in \mathcal{R}$ if and only if $b_i$ is not sincere for $R_i$, that is, if and only if there exist $x, y \in X$ such that

$$[b_i(x) \geq b_i(y) \text{ and } y P_i x] \text{ or } [b_i(x) > b_i(y) \text{ and } y R_i x].$$

3 Richness

An important reason to consider voting by ballots with vote budgets is to enable a voter $i \in N$ to express any possible preference ordering $R_i \in \mathcal{R}$ by means of a sincere ballot $b_i \in B(v)$. Clearly, the plurality rule (Richelson, 1978; Ching, 1996) fails to satisfy this requirement if there are at least three candidates—the only options available to a voter are to cast a single vote in favor of one candidate, or to abstain. Thus, if there is a tie for first place with other alternatives ranked below the top candidates, such a preference cannot be expressed by means of a sincere ballot. In general, no preference ordering that differs from universal indifference and from the dichotomous preferences with a unique top element can be adequately represented. Approval voting (Brams and Fishburn, 1978) fares better with respect to this criterion but still does not allow all possible preferences to be expressed sincerely. By definition, approval voting is only capable of producing sincere ballots if there is universal indifference or preferences are dichotomous (but not necessarily with a single top element). Nitzan et al. (1980) also examine the possibilities for preference expression in their contribution. However, because they allow votes to be perfectly divisible, it is no surprise that their points-voting method imposes no restriction whatsoever on the set of preferences that can be represented. This is in stark contrast with the vote-budget
procedures analysed here and, therefore, their observations do not apply. We reiterate that restricting ballots to integer-valued votes is of considerable practical relevance.

To guarantee that the voters’ preferences can be adequately represented by the requisite voting rule, the vote budget \( v \in \mathbb{N} \) must satisfy the following richness condition.

**Richness.** For every \( R_i \in \mathcal{R} \), there exists a sincere ballot \( b_i \in B(v) \).

The values of the vote budget that satisfy this richness axiom are identified in the following result.

**Theorem 1** A vote budget \( v \in \mathbb{N} \) satisfies richness if and only if \( v \geq (m - 1) \cdot m/2 \).

**Proof.** Suppose first that \( v \in \mathbb{N} \) satisfies richness. Without loss of generality, suppose that \( R_i \) is such that

\[
x_m R_i \ldots R_i x_1.  
\]

Richness requires that a sincere ballot \( b_i \in B(v) \) satisfies the inequalities

\[
b_i(x_m) \geq \ldots \geq b_i(x_1).
\]

Because the votes appearing on a ballot must be non-negative integers, it follows that \( b_i(x_1) \geq 0 \) and, furthermore,

\[
b_i(x_k) = b_i(x_{k-1})
\]

for all \( k \in \{2, \ldots, m\} \) such that \( x_k I_i x_{k-1} \), and

\[
b_i(x_k) \geq b_i(x_{k-1}) + 1
\]

for all \( k \in \{2, \ldots, m\} \) such that \( x_k P_i x_{k-1} \). Clearly, the most demanding case (demanding in the sense of requiring the largest vote budget to ensure that the requisite preference ordering can be expressed sincerely) is that in which all preferences in (1) are strict. As a consequence, all inequalities in (2) must be strict and, together with the integer requirement, it follows that (3) must be true for all \( k \in \{2, \ldots, m\} \). Iterative application of these inequalities yields

\[
b_i(x_k) \geq b_i(x_1) + k - 1
\]

and, because \( b_i(x_1) \geq 0 \) by the non-negativity requirement for the votes to be cast, we obtain

\[
b_i(x_k) \geq k - 1
\]

for all \( k \in \{2, \ldots, m\} \). Adding over all values of \( k \), it follows that the vote budget must satisfy

\[
v \geq \sum_{k=1}^{m} b_i(x_k) = \sum_{k=2}^{m} b_i(x_k) + b_i(x_1) \geq \sum_{k=2}^{m} (k - 1) + 0 = \sum_{k=1}^{m-1} k = \frac{(m - 1) \cdot m}{2},
\]

as was to be established.

Conversely, it follows immediately that any vote budget \( v \) such that

\[
v \geq \frac{(m - 1) \cdot m}{2}
\]

satisfies richness because, as illustrated in the above argument, any such \( v \) accommodates the most vote-intensive scenario in which all preferences are strict. ■
4 Strategic considerations

Theorem 1 provides a lower bound on $v$ in order to ensure that a sincere ballot can be found for every possible ordering of the candidates. A force pulling in the opposite direction is the potential for individual manipulation by casting insincere ballots for some orderings on $X$. Because the sets of candidates chosen by the correspondence $C_v$ may contain more than one element, it is necessary to introduce a method by which a voter can carry out comparisons of such sets (as opposed to be merely able to rank single candidates). For each $i \in N$, let $\succsim_i(R_i)$ be a relation on $X$ (with associated strict preference relation $\succ_i(R_i)$) that may depend on the ordering $R_i$ on $X$. Note that no assumptions have to be made regarding the properties of this relation. In particular, it may be consistent with any of the non-probabilistic approaches such as those of Gärdenfors (1976) and Kelly (1977) as well as the expected-utility criterion employed by Barberà et al. (2001).

A voting rule $C_v$ is manipulable at $b \in B(v)^n$ and $R \in \mathcal{R}^n$ by a voter $j \in N$ via an insincere ballot $b'_j \in B(v)$ if it is the case that

$$C_v(b_{-j}, b'_j) \succ_j(R_j) C_v(b)$$

where $b_{-j} = (b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_n)$. Clearly, whenever $v < v'$ for two possible vote budgets $v, v' \in \mathbb{N}$, the set $B(v)$ is a subset of the set $B(v')$ and, as a consequence, it follows that $B(v)^n \subset B(v')^n$. Therefore, if $C_v$ is manipulable at $b \in B(v)^n$ and $R \in \mathcal{R}^n$ by $j \in N$ via $b'_j$, it follows immediately that $C_{v'}$ is manipulable at $b \in B(v')^n$ and $R \in \mathcal{R}^n$ by $j$ via $b'_j$. Thus, increasing the value of $v$ can only make the voting rule $C_v$ more susceptible to strategic manipulation. This observation, coupled with Theorem 1, leads us to the following result.

**Theorem 2** The voting rule $C_{v^*}$ is least susceptible to strategic manipulation among the voting rules by ballots with vote budgets that satisfy richness if

$$v^* = \frac{(m-1) \cdot m}{2}.$$

The statement of this theorem does not claim that an increase in the vote budget always strictly expands the set of profiles at which strategic manipulation can occur. It seems that observations of such a general nature are difficult to establish in this setting. Note that, for instance, we do not impose any restrictions on the relations $\succsim_i(R_i)$ that allow voters to compare election outcomes composed of several candidates and, as a consequence, there is bound to be some indeterminacy.

5 Discussion

As mentioned earlier, the plurality rule and approval voting do not satisfy the richness requirement introduced earlier. However, there remains the question of whether a voter has the option of casting ballots in accordance with those proposed for these choice rules.
This is indeed the case: plurality voting requires a vote budget of one, whereas approval voting can be defined as long as the vote budget $v$ is greater than or equal to $m - 1$ which, for any value of $m$ greater than or equal to two, is less than or equal to $(m - 1) \cdot m/2$.

A more subtle case is that of the Borda count. There are several ways of expressing this voting rule; what is common to them is that all these methods of computing the Borda scores are increasing affine transformations of each other. Two examples are given in the original contribution by Borda (1781); see also Black (1958, pp. 156–159). According to the first method, the top candidate receives a score of $m$, the second-best a score of $m - 1$ and so on until the bottom candidate is reached with a score of one. This procedure can easily be generalized to accommodate indifference by assigning the average score to any candidates in the same indifference class. The second procedure uses the number of candidates beaten by a candidate $x$ in a pairwise comparison minus the number of candidates who beat $x$ in a pairwise comparison as the score assigned to alternative $x$. The two functions that assign Borda scores to the candidates are increasing affine transformations of each other, which is the reason why either of them can be used in determining the corresponding voting rule: selecting the candidates with the highest total Borda scores leads to the same set of chosen candidates for any two methods of assigning scores that are increasing affine transformations of each other.

In the context of voting by ballots with vote budgets, the question arises whether assigning votes according to the Borda method is compatible with the vote budget $v^* = (m - 1) \cdot m/2$ identified in Theorem 2. Clearly, the two variants alluded to above cannot be employed: the first of these requires $1 + \ldots + m = m \cdot (m + 1)/2$ votes which exceeds $v^*$; the second requires negative numbers to be assigned to some candidates. An immediate remedy that can be applied to the first assignment rule is to use the numbers $\{0, \ldots, m - 1\}$ in place of $\{1, \ldots, m\}$ so that we comply with the vote budget given by $v^*$. However, such a move may lead to a conflict with the integer requirement if indifferences are present, as established by the following simple example. If $X = \{x_1, x_2, x_3\}$ and $x_3 I_1 x_2 P_1 x_1$, the scores assigned to the three candidates are zero to candidate $x_1$ and $3/2$ to each of candidates $x_2$ and $x_3$.

But there may still be other equivalent Borda-score assignment rules that are compatible with our setting. Clearly, if such methods exist, they have to be increasing affine transformations of the original rule. To examine this possibility, suppose that, without loss of generality, $x_m R_1 \ldots R_1 x_1$ and consider the Borda scores

$$s_i(x_k) = k - 1$$

for all $k \in \{1, \ldots, m\}$ with averages assigned in the case of indifference. Now suppose that there are parameters $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}$ such that the individual scores are given by

$$b_i(x_k) = \alpha s_i(x_k) + \beta$$

for all $k \in \{1, \ldots, m\}$. The question to be addressed now is whether values of $\alpha$ and $\beta$ can be found so that all these scores are consistent with (i) the non-negativity requirement

$$b_i(x_k) \geq 0$$
for all $k \in \{1, \ldots, m\}$; (ii) the vote-budget constraint
\[\sum_{k=1}^{m} b_i(x_k) \leq v^* = \frac{(m - 1) \cdot m}{2};\]
and (iii) the demand that each score $b_i(x_k)$ be an integer. Note that these conditions must be met for any possible preference ordering $R_i \in \mathcal{R}$.

Observe first that if $R_i$ is such that $x_2 P_i x_1$, we have
\[b_i(x_1) = \alpha s_i(x_1) + \beta = \alpha \cdot 0 + \beta = \beta\]
and the integer requirement immediately implies that $\beta$ must be an integer.

The total number of votes required to sincerely express a strict preference relation is given by
\[\sum_{k=1}^{m} b_i(x_k) = \alpha \sum_{k=1}^{m} s_i(x_k) + m\beta = \alpha \sum_{k=1}^{m} (k - 1) + m\beta = \alpha \frac{(m - 1) \cdot m}{2} + m\beta\]
and the vote-budget constraint demands that
\[\alpha \frac{(m - 1) \cdot m}{2} + m\beta \leq \frac{(m - 1) \cdot m}{2}\]
which, because $\beta$ is non-negative, implies that $\alpha \leq 1$.

Now suppose that $R_i$ is such that $x_2 P_i x_1$ and, if $m \geq 3$, all other candidates are strictly preferred to $x_2$. It follows that $b_i(x_2) = \alpha/2 + \beta$. Because $\beta$ is an integer, $\alpha/2$ must be an integer to ensure that $b_i(x_2)$ is an integer. This implies that $\alpha$ is a positive multiple of two, which is impossible because $\alpha \leq 1$ as was just established. Thus, there is no representation of the Borda scoring method that is compatible with the requirements of non-negativity, richness and integer-valuedness of the votes.

Another voting procedure consists of what is sometimes referred to as range voting or evaluative voting; see, for instance, Baujard et al. (2014) and some of the references cited there. In its standard form, range voting proceeds as follows. Each voter can assign a number of votes between 0 and a given maximal value $u \in \mathbb{N}$ to each candidate. Specifically, the set of feasible ballots for voter $i \in \mathbb{N}$ with a maximal value $u \in \mathbb{N}$ is defined by
\[B(u) = \{b_i = (b_i(x_1), \ldots, b_i(x_m)) \in \mathbb{N}_{0}^m \mid 0 \leq b_i(x_k) \leq u \text{ for all } k \in \{1, \ldots, m\}\}.
As is the case for voting by means of vote budgets, the most demanding (in the sense of requiring the highest maximal value) is a situation in which all preferences in (1) are strict. Thus, in order to satisfy the richness requirement, $u$ must be such that we can assign zero votes to candidate $x_1$, one vote to candidate $x_2$ and so on until we reach $m - 1$ votes to be given to candidate $x_m$. Thus, the maximal value $u$ must satisfy the inequality $u \geq m - 1$. Because the maximal value $u$ must be the same for each candidate, it follows that the total number of votes available to a voter is $(m - 1) \cdot m$—which is twice as high as the corresponding lower bound for the procedure based on Dodgson’s method of marks.
It is essential to assume that there is a non-negativity constraint and an integer requirement when discussing the richness property in the context of voting rules that employ some form of vote budget. Without non-negativity, the second variant of the Borda rule mentioned earlier trivially satisfies any richness requirement because, in that case, the requisite Borda scores sum to zero. Without the integer requirement, a conclusion analogous to that of Nitzan et al. (1980) alluded to in the introduction for the case of divisible votes emerges.

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